

# The beta function of a knot

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## 0. Introduction

In this note we study a metric invariant of a knot  $K$  embedded in  $\mathbb{R}^3$ . This metric invariant depends on a complex parameter  $s$ . It is defined as a double integral using the parameterization  $x \mapsto \gamma(x)$  of  $K$  by the arc-length parameter  $x$ :

$$B_K(s) = \int_{K \times K} \|\gamma(x) - \gamma(y)\|^s dx dy \quad (1)$$

where  $dx, dy$  denote the arc-lengths on the two copies of  $K$ . Clearly the function  $(x, y) \mapsto \|\gamma(x) - \gamma(y)\|^s$  is continuous for  $\operatorname{Re}(s) > 0$ , so that the integral converges in that domain. One main result is

**Theorem 0.1.** *For a smooth knot  $K$ , the function  $s \mapsto B_K(s)$  extends analytically to a meromorphic function on  $\mathbb{C}$ , with only possible poles at  $-1, -3, -5, \dots$ . The residue at  $-2j - 1$  is of the form  $\int_K P_j(\kappa, \tau) dx$ , where  $P_j$  is an explicitly computable polynomial in the curvature  $\kappa$ , the torsion  $\tau$  and their derivatives.*

For  $s = -1, -3, -5$  the polynomial  $P_j$  is described in §3.

The case  $s = -2$  is  $E(K) - 4$ , where  $E(K)$  is the Möbius energy of  $K$  studied in [F-H-W] [O] [K-K] [K-S].

The theorem is proved in section 3.

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## 1. The case of the circular knot.

We give the computation of  $B_K(s)$  for the trivial circular knot. In this case we have, using the rotational symmetry:

$$\begin{aligned} B_K(s) &= 2\pi \int_0^{2\pi} [2 \sin(\frac{x}{2})]^s dx \\ &= 2^{s+2} \pi \int_0^\pi \sin(x)^s dx . \\ &= 2^{s+3} \pi \int_0^{\frac{\pi}{2}} \sin(x)^s dx \end{aligned}$$

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The integral is classically evaluated explicitly, via the change of variable  $\sin(x) = \sqrt{y}$ . We have:

$$\int_0^{\frac{\pi}{2}} \sin(x)^s dx = \frac{1}{2} \int_0^1 y^{\frac{s-1}{2}} (1-y)^{\frac{-1}{2}} dy = B\left(\frac{s}{2} + \frac{1}{2}, \frac{1}{2}\right),$$

using the Eulerian *beta function*

$$B(p, q) = \int_0^1 x^{p-1} (1-x)^{q-1} dx$$

which is equal to  $\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$  **[W-W]**. Therefore we find

$$\int_0^{\frac{\pi}{2}} |\sin(x)|^s dx = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{(s+1)}{2})}{2\Gamma(\frac{s}{2} + 1)}$$

This is of course a meromorphic function of  $s$ , and its poles are located precisely at the poles of  $\Gamma(\frac{(s+1)}{2})$ , which occur exactly when  $\frac{(s+1)}{2}$  belongs to  $\{0, -1, -2, \dots\}$ , i.e., when  $s \in \{-1, -3, \dots\}$ . This illustrates the theorem, and it also justifies calling  $B_K(s)$  the *beta function* of the knot  $K$ .

The functional equation for the gamma function immediately gives the functional equation

$$B_K(s) = 4 \frac{s-1}{s} B_K(s-2)$$

which can also be verified directly from the integral expression for  $B_K(s)$ , using integration by parts.

## 2. The case of polygonal knots

We consider a polygonal knot  $K$ , with  $n$  sides  $K_1, \dots, K_n$  of lengths  $a_1, \dots, a_n$ . It is convenient to think of the set of edges as  $\mathbb{Z}/n\mathbb{Z}$ . We denote by  $v_j$  the vertex belonging to the sides  $K_j$  and  $K_{j+1}$  (so  $v_n$  belongs to  $K_n$  and  $K_1$ ). Let  $\phi_j \in (0, \pi)$  be the angle between these two edges. We study the function  $B_K(s)$  by decomposing it into  $\frac{n+1}{2}$  double integrals over  $K_j \times K_l$ , where  $j < l$ . If  $K_j$  and  $K_l$  do not meet, then clearly the double integral  $\int_{K_j \times K_l} \|\gamma(x) - \gamma(y)\|^s dx dy$  extends to a holomorphic function in  $\mathbb{C}$ . So we have to consider two types of double integrals:

$$A_j = \int_{K_j \times K_j} \|\gamma(x) - \gamma(y)\|^s dx dy$$

and

$$B_j = \int_{K_j \times K_{j+1}} \|\gamma(x) - \gamma(y)\|^s dx dy.$$

We have

$$A_j = \int_0^{a_j} \int_0^{a_j} |x - y|^s dx dy.$$

We evaluate this as follows: first we fix  $x$  and integrate over  $y$ , which yields the expression  $\frac{1}{s+1}[x^{s+1} + (a_j - x)^{s+1}]$ . So we have

$$A_j = \frac{2}{s+1} \int_0^{a_j} x^{s+1} dx = \frac{2}{(s+1)(s+2)} a_j^{s+2},$$

which is meromorphic with simple poles at  $s = -1$  and  $s = -2$ . The residues are  $2a_j^{-1}$  and  $-2$  respectively.

Now we study  $B_j$ . If  $u$  resp.  $v$  is the unit vector pointing from  $v_j$  to  $v_{j-1}$  (resp. from  $v_j$  to  $v_{j+1}$ ), we have:

$$B_j = \int_0^{a_j} \int_0^{a_{j+1}} ||au - bv||^s da db.$$

This is simplified somewhat by introducing the plane parallelogram  $P$  spanned by the vectors  $a_j u$  and  $-a_{j+1} v$ . We have in polar coordinates in the plane of  $P$ :

$$B_j = \int_P \sin(\phi_j)^{-1} r^s r dr d\theta.$$

The factor  $\sin(\phi_j)^{-1}$  appears here as the jacobian of the change of variables from  $(a, b)$  to cartesian coordinates  $(x, y)$  in the plane of  $P$ . Since we wish to determine whether this function of  $s$  has a meromorphic extension and where the poles are located, we do not need to evaluate explicitly, in the sense that we may subtract the integral over any subregion which is clearly holomorphic in  $\mathbb{C}$ . Thus we may replace the parallelogram  $P$  by its intersection with a disc of radius  $\epsilon$  centered at 0, to get the function

$$\begin{aligned} C_j &= \sin(\phi_j)^{-1} \int_0^\epsilon \int_0^{\pi-\phi_j} r^s r dr d\theta \\ &= \sin(\phi_j)^{-1} (\pi - \phi_j) \frac{1}{s+2} [r^{s+2}]_0^\epsilon \end{aligned}$$

which for  $Re(s) > -2$  is equal to

$$\frac{1}{s+2} \epsilon^{s+2} \sin(\phi_j)^{-1} (\pi - \phi_j)$$

This makes it clear that  $C_j$  is a meromorphic function of  $s$  with only a simple pole at  $s = -2$ , with residue equal to

$$\frac{\pi - \phi_j}{\sin(\phi_j)}.$$

Now  $B_K(s) - \sum A_j - 2 \sum C_j$  is an entire function. Putting all this information together, we obtain

**Proposition 2.1.** (1) For a polygonal knot  $K$ , the beta function  $B_K(s)$  extends to meromorphic function of  $s$ , with at most simple poles at  $s = -1$  and  $s = -2$ .

(2) We have:

$$\text{Res}_{s=-1} B_K(s) = l(K), \text{Res}_{s=-2} B_K(s) = -2n + 2 \sum_{j=1}^n \frac{\pi - \phi_j}{\sin(\phi_j)}$$

where  $l(K)$  is the length of  $K$ ,  $n$  is the number of sides, and the  $\phi_j$  are the angles between the two edges at the vertices  $v_1, \dots, v_n$ . The residues are always  $> 0$ .

The last statement follows from the fact that  $\pi - \phi_j > \sin(\phi_j)$ . We note that there is no harm in introducing a fictitious vertex by splitting an edge at some interior point. We note indeed that as  $\phi$  tends to  $\pi$  the function  $\frac{\pi - \phi}{\sin(\phi)}$  tends to 1. Then the residue at  $s = -2$  is unchanged by this operation, as it must be.

### 3. The beta function of a smooth knot.

Clearly the problem with the double integral (1-1) is that it will diverge near the diagonal  $x = y$ . We write down the Taylor series for the parameterization  $x \mapsto \gamma(x)$  of  $K$  by the arc-length parameter  $x$ :

$$\gamma(y) - \gamma(x) = \sum_{j=1}^r \frac{1}{j!} (y - x)^j \gamma^{(j)}(x) + (y - x)^{r+1} \alpha_r(x, y), \quad (3-1)$$

where  $\alpha_r$  is a smooth vector-valued function of  $(x, y)$ . For the purpose of studying the analytic properties of  $B_K(s)$ , we may replace it by the function

$$D(s) = \int_{|x-y| \leq \epsilon} \|\gamma(x) - \gamma(y)\|^s dx dy \quad (3-2)$$

for some small  $\epsilon$ , which will be determined later. For the square of the distance function we have an expression of the type:

$$\|\gamma(x) - \gamma(y)\|^2 = |y - x|^2 \left( 1 + \sum_{i=1}^{r-1} f_i(x) (y - x)^i + (y - x)^r Q(x, y) \right),$$

where each  $f_i$  is a smooth function of  $x$ , and  $Q(x, y)$  is also smooth. When we take the  $\frac{s}{2}$ -th power, we use the power series expansion  $(1 + a)^{\frac{s}{2}} = \sum_{j \geq 0} \binom{\frac{s}{2}}{j} a^j$  which has radius of convergence 1, and gives a holomorphic function of  $(a, s)$  in the domain  $s \in \mathbb{C}, |a| < 1$ . We now assume that  $\epsilon$  is picked small enough so that

$|\sum_{i=1}^{r-1} f_i(x)(y-x)^i + (y-x)^r Q(x,y)| < 1$  whenever  $|y-x| < \epsilon$ . Then in that region we obtain an expansion:

$$\|\gamma(x) - \gamma(y)\|^s = |y-x|^s \left(1 + \sum_{i=1}^{r-1} h_i(s,x)(y-x)^i + (y-x)^r R(s,x,y)\right),$$

where  $h_i(s,x)$  is a smooth function of  $(s,x)$  which is polynomial as a function of  $s$ , and  $R(s,x,y)$  is a smooth function of  $(s,x,y)$  which is an entire function of  $s$ . We then have

$$D(s) = \int_0^l \int_{-\epsilon}^{\epsilon} \left[|y-x|^s + \sum_{i=1}^{r-1} h_i(s,x)|y-x|^s(y-x)^i + R(s,x,y)|y-x|^s(y-x)^r\right] dx dy \quad (3-3)$$

In the region  $Re(s) > -r$  the last term of the integral is holomorphic, so the behaviour of  $D(s)$  can be inferred from the other terms. The integral over  $y$  is easily evaluated, and it is clear that for  $i$  odd it is equal to 0, whereas for  $i = 2$  even we have

$$\int_{-\epsilon}^{\epsilon} |y-x|^{s+2j} dy = \frac{2}{s+2j+1} \epsilon^{s+2j+1}.$$

This is meromorphic in  $\mathbb{C}$ , with a simple pole at  $s = -2j-1$  with residue equal to 2. It then follows that  $D(s)$  is meromorphic in the region  $Re(s) > -r$  with possible poles at  $-1, -3, \dots, -r+1$ , with residue at  $-2j-1$  equal to  $2 \int_0^l h_j(-2j-1, x) dx$ . Since this is true for all  $r \geq 1$ , Theorem 0.1 is proved. The fact that  $h_j(s,x)$  has a differential expression in terms of  $\kappa$  and  $\tau$  follows since any scalar product  $\gamma^{(p)}(x) \cdot \gamma^{(q)}(x)$  of derivatives of  $\gamma$  has such an expression.

We now compute the residues at  $s = -1$  and  $s = -3$ . We easily obtain the following expression for the square of the distance function:

$$\begin{aligned} \|\gamma(x) - \gamma(y)\|^2 &= (y-x)^2 \left(1 + (y-x)^2 \left(\frac{1}{8} \|\gamma''(x)\|^2 + \frac{1}{3} \gamma'(x) \cdot \gamma'''(x) + O(y-x)^4\right)\right) \\ &= (y-x)^2 \left(1 - \frac{5}{24} (y-x)^2 \|\gamma''(x)\|^2 + O(y-x)^4\right) \end{aligned}$$

using the fact that  $\gamma' \cdot \gamma'' = 0$  and the corollary that  $\gamma' \cdot \gamma''' + \gamma'' \cdot \gamma'' = 0$ . We then have

$$\|\gamma(x) - \gamma(y)\|^s = |y-x|^s \left(1 - \frac{5s}{48} \|\gamma''(x)\|^2 (y-x)^2 + O(y-x)^4\right).$$

We then derive the formula of the first two residues:

$$Res_{s=-1} B_K(s) = 2l = 2l(K), \quad Res_{s=-3} B_K(s) = \frac{5}{8} \int_K \kappa^2 dx.$$

The residue at  $-5$  can be similarly calculated. It is equal to  $\int_K Q dx$ , where

$$Q = \frac{(\kappa')^2}{8} - \frac{\kappa^2 \tau^2}{144} + \frac{859}{16 \times 144} \kappa^4.$$

We now compare the value  $B_K(-2)$  of the beta function with the Möbius energy  $E(K)$  studied by O'Hara [O] and by Freedman, He and Wang [F-H-W]. This energy is defined as

$$E(K) = \int_{K \times K} [ \|\gamma(x) - \gamma(y)\|^{-2} - d(x, y)^{-2} ] dx dy,$$

where  $d(x, y) = \int_x^y ds$  is the distance between  $x$  and  $y$  along the knot.

This functional is the value at  $s = -2$  of the following functional

$$D_K(s) = \int_{K \times K} [ \|\gamma(x) - \gamma(y)\|^s - d(x, y)^s ] dx dy.$$

This function of  $s$  is clearly holomorphic in the region  $Re(s) > -3$  from the asymptotic expansion of  $\|\gamma(x) - \gamma(y)\|^s$  previously described. The function

$$f(s) = \int_{K \times K} d(x, y)^s dx dy$$

is easily evaluated to be equal to  $\frac{2^{-s} l^{s+2}}{s+1}$ , which is holomorphic near  $s = -2$  with value at  $s = -2$  equal to 4.

Hence we find

**Proposition 3.1.** *The functional  $B_K(-2)$  is equal to  $E(K) - 4$ .*

In particular, the functional  $B_K(-2)$  coincides with the version of the Möbius energy given in [K-K], which is normalized to vanish for the circle knot. J. Sullivan pointed out to me that this vanishing, in the case of  $B_K(-2)$ , follows immediately from the functional equation given in section 1.

## 4. Poisson brackets

In [Br] we introduced a symplectic structure over the Fréchet manifold  $Y$  of all smooth knots in  $\mathbb{R}^3$ . This induces a Poisson bracket over a class of smooth functionals  $f$ , called supersmooth. This class of functionals  $f$  is such that the differential  $df$  at a knot  $K$  can be viewed as a section of the restriction to  $K$  of the cotangent bundle of  $\mathbb{R}^3$ , which we may view as a vector field to  $\mathbb{R}^3$  defined along  $K$ . This vector field is denoted by  $df$ . We then have

$$\{f, g\}_K = -l(K)^{-2} \int_K \det\left(\frac{d\gamma}{dx}, df(x), dg(x)\right) dx$$

[Br, (3-11), p. 138]. We apply this to two functionals  $K \mapsto B_K(s)$  and  $K \mapsto B_K(u)$ . These two functionals will be denoted by  $B_s$  and  $B_u$ . We first assume that the real parts of  $s$  and  $u$  are large enough. This will allow us to compute as many derivatives of the functional  $K \mapsto B_s(K)$  as needed. We see that the derivative  $dB_s$  at a knot  $K$  is given by the vector-valued function on  $K$ :

$$[dB_s]_K(x) = -2s \int_K \|\gamma(x) - \gamma(y)\|^{s-1} (\gamma(x) - \gamma(y)) dy.$$

Therefore the Poisson bracket is defined and given by the triple integral

$$\{B_s, B_u\}_K = 4sul(K)^{-2} \int_{K \times K \times K} \|\gamma(x) - \gamma(y)\|^{s-1} \|\gamma(x) - \gamma(z)\|^{u-1} \det\left(\frac{d\gamma}{dx}, \gamma(x) - \gamma(y), \gamma(x) - \gamma(z)\right) dx dy dz.$$

Now, while keeping  $Re(u)$  large, we want to move  $s$  to a vicinity of  $s = -1$ . We should then expand the integrand in powers of  $y - x$ . Since we have

$$\det\left(\frac{d\gamma}{dx}, \gamma(x) - \gamma(y), \gamma(x) - \gamma(z)\right) = (x - y)^2 \det\left(\frac{d\gamma}{dx}, \gamma''(x), \gamma(x) - \gamma(z)\right) + O(x - y)^3$$

in a neighborhood of the partial diagonal  $y = x$ , it is clear that the integrand is of the form  $|y - x|^{s+1}$  times a continuous function. Therefore the Poisson bracket  $\{B_s, B_u\}$  has no singularity at  $s = -1$ . This implies

**Proposition 4.1.** *The length functional  $l$  on the space of knots  $Y$  Poisson commutes with all the functionals  $u \mapsto B_K(u)$ .*

**Corollary 4.2.** *For any  $j \geq 1$ , the length functional on  $Y$  Poisson commutes with the functional  $K \mapsto Res_{s=-2j-1} B_K(s)$ .*

The fact that  $l$  Poisson commutes with the functional  $\int_K \kappa^2 dx$  is well-known (see [Br]).

## 5. A Bernstein-Sato type equation.

For a polynomial  $f(z_1, \dots, z_n)$ , J. Bernstein [Be] proved that there exists a partial differential equation

$$P(\underline{z}, \frac{\partial}{\partial \underline{z}}, s) f^s = B(s) f^{s-1}, \quad (5-1)$$

where  $P$  is a differential operator with polynomial coefficients, which depends polynomially in the variable  $s$ . Here  $B(s)$  is a monic polynomial; when it is chosen of

minimal degree, it is called the Bernstein-Sato polynomial of  $f$  and the equation (5-1) is called a Bernstein-Sato equation. In some cases the equation was known for quite some time: for instance, if  $f(z_1, \dots, z_n) = z_1^2 + \dots + z_n^2$  and if we set  $P(\underline{z}, \frac{\partial}{\partial \underline{z}}) = \frac{1}{4} \sum_{i=1}^n \frac{\partial^2}{\partial z_i^2}$ , then we have

$$Pf^s = s(s-1 + \frac{n}{2})f^{s-1}.$$

This equation can be found in the treatise of Gelfand and Shilov [G-S].

We will establish an analog of this equation for the functional  $B_K(s)$  on the space of knots  $Y$ . First we write down an analog of the Laplace operator. Given a knot  $K$  and 2 points of  $K$  with arc-length parameters  $x, y$ , we introduce the unit vector

$$w(x, y) = \gamma(x) - \gamma(y) / \|\gamma(x) - \gamma(y)\|.$$

This is of course a smooth vector-valued function on  $K \times K$ .

Now let  $f$  be a supersmooth functional on  $Y$ . Then its second order differential  $D^2f$ , evaluated at  $K$ , is an  $\mathbb{R}^3 \otimes \mathbb{R}^3$ -valued function on  $K \times K$ . We set

$$P(f)(K) = \int_{K \times K} \langle D^2f(x, y), w(x, y) \otimes w(x, y) \rangle dx dy \quad (5-2)$$

We then have

**Proposition 5.1.** *For  $Re(s) \gg 0$ , we have:*

$$PB_s = s(s-1)B_{s-2} \quad (5-3)$$

**Proof.** This follows immediately from the fact that the second derivative of  $\|\gamma(x) - \gamma(y)\|^s$  evaluated at  $(x, y)$  on the element  $w \otimes w$  is equal to  $s(s-1)\|\gamma(x) - \gamma(y)\|^{s-2}$ . ■

## REFERENCES

- [Be] J. Bernstein, *The analytic continuation of generalized functions with respect to a parameter*, Funct. Anal. Appl. **6**(1972), 272-285
- [Br] J-L. Brylinski, *Loop Spaces, Characteristic Classes and Geometric Quantization*, Progress in Math. vol. **107** (1993), Birkhäuser
- [F-H-W] M.H. Freedman, Z.X. He and Z. Wang, *Möbius energy of knots and unknots*, Annals of Math **139** (1994), 1-50
- [G-S] I. M. Gel'fand and G. E. Shilov, *Generalized Functions*, Academic Press
- [K-K] D. Kim and R. Kusner, *Torus knots extremizing the Möbius energy*, Experiment. Math. **2**(1993), 1-9

[**K-S**] R. B. Kusner and J. M. Sullivan, *Möbius Energies for Knots and Links, Surfaces and Submanifolds*, in Geometric Topology, ed. W. H. Kazez, Amer. Math. Soc./Int'l Press (1997), 570-604

[**O**] J. O'Hara, *Energy of a Knot*, Topology **30** (1991), 241-247

[**W-W**] E. T. Whittaker and G. N. Watson, *A course in Modern Analysis*, Cambridge Univ. Press

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